1. This question is about the equation

$$c\phi + \boldsymbol{u} \cdot \nabla\phi - \epsilon\nabla^2\phi = f \text{ on } \Omega, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where:

- Ω is a d-dimensional polygonal domain with boundary $\partial\Omega$,
- $c > 0$,
- f is a known function,
- $\boldsymbol{u} \in C^{1,\infty}(\Omega)^d$ is a known vector-valued function satisfying $\nabla \cdot \boldsymbol{u} = 0$, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.
- $|\boldsymbol{u}|_\infty = \max_{\boldsymbol{x}\in\Omega} |\boldsymbol{u}(\boldsymbol{x})| = C_0 < \infty$.

(a) Derive a weak formulation of this equation for a solution $\phi \in H^1(\Omega)$ of the form

$$a(q, \phi) = F(\phi), \quad \forall H^1(\Omega). \tag{2}$$

[5 marks]

(b) Obtain estimates for the continuity and coercivity constants of $a(\cdot, \cdot)$.

[10 marks]

(c) What happens to the H^1 norm of the error in the P^1 finite element approximation of this problem as $\epsilon \to 0$? Justify your answer.

[5 marks]

2. (a) For a ball B in a triangle K, the averaged Taylor polynomial of a function $u \in H^k(K)$ of degree k is defined by

$$Q_{k,B}u(x) = \frac{1}{|B|}\int_B \sum_{|\alpha|\leq k} D^\alpha u(y)\frac{(x-y)^\alpha}{\alpha!}\,\mathrm{d}\,y. \tag{3}$$

For $|\beta| \leq k$ show that

$$D^\beta Q_{k,B}u(\boldsymbol{x}) = Q_{k-|\beta|,B}D^\beta u(\boldsymbol{x}). \tag{4}$$

[8 marks]

(b) For the rest of the question we assume that K has radius 1. Let $u \in H^{k+1}(K)$. Assuming that, for $i \leq k$,

$$\|Q_{i,B}u - u\|_{L^2(K)} \leq C|u|_{H^{k+1}(K)}, \tag{5}$$

show that

$$\|D^\beta(Q_{i,B}u - u)\|_{L^2(K)} \leq C|u|_{H^{k+1}(K)}, \tag{6}$$

for $|\beta| \leq i \leq k$.

[8 marks]

(c) Using the property

$$\|I_K u\|_{H^k(K)} \leq C_1\|u\|_{H^k(K)}, \tag{7}$$

for the nodal interpolation operator I_K corresponding to a finite element $(K, \mathcal{P}, \mathcal{N})$, show that

$$|I_K u - u|_{H^k(K)} \leq C_2|u|_{H^{k+1}(K)}, \tag{8}$$

for some positive constant C_2, stating any assumptions you make about $(K, \mathcal{P}, \mathcal{N})$.

[4 marks]

3. Consider the following triple $(K, \mathcal{P}, \mathcal{N})$.

- K is a triangle with vertices z_1, z_2, z_3.
- $\mathcal{P}$ are the polynomials of degree ≤ 3.
- $\mathcal{N}$ are dual variables given by evaluations at $z_1+(z_2-z_1)i/3+(z_3-z_1)j/3$ for $0 \leq i \leq j \leq 3$.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$.

[10 marks]

(b) Describe the geometric decomposition for this finite element, and explain why it is a C^0 decomposition.

[10 marks]

4. Consider the heat equation,

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T, \tag{9}$$

solved for a time-dependent function T on a closed simply-connected domain Ω, with boundary conditions $\frac{\partial T}{\partial n} = 0$ on the boundary $\partial\Omega$.

(a) Given a C^0 finite element space, formulate a finite element discretisation of the heat equation (9).

[5 marks]

(b) Show that the discretisation can be written in the form

$$M\dot{\boldsymbol{T}} = K\boldsymbol{T}, \tag{10}$$

where $\boldsymbol{T}$ is the vector of basis coefficients for T in the finite element space V_h.

[5 marks]

(c) Quoting results from lectures, show that

$$\frac{d}{dt}\int_\Omega T^2 \,\mathrm{d}\,x \leq -C\int_\Omega T^2 \,\mathrm{d}\,x, \tag{11}$$

providing an upper bound for the decay rate C.

[5 marks]

(d) Explain why this means that the decay rate for the finite element discretisation is larger than or equal to the decay rate for the unapproximated equation.

[5 marks]

5. This question is based upon the Mastery material "From Functional Analysis to Iterative Methods" by RC Kirby.
Consider the partial differential equation

$$-\nabla \cdot (\gamma(x)\nabla u) = f, \tag{12}$$

on Ω, with boundary conditions $u = 0$ on $\partial\Omega$, where f is a known function with $\|f\|_{L^2(\Omega)} < \infty$, and γ is a known function with $c_1 \leq \gamma \leq c_2$ for $c_1 > 0$, $c_2 < \infty$.

(a) Briefly formulate a finite element discretisation for this problem using linear continuous finite elements, and explain how the coercivity and continuity constants of the variational problem depend on c_1 and c_2. Give details on the function spaces involved and norms involved.

[6 marks]

(b) A bilinear form a on a finite element space V_h defines an operator $A_h : V_h \to V_h'$ into the dual space given by

$$(A_h f)[u] = a(f, u), \quad \forall f, u \in V_h. \tag{13}$$

In the notation of the paper, the operator $\mathcal{I}_h : \mathbb{R}^{\dim V_h} \to V_h$ maps a vector to the function in V_h with the vector entries as basis coefficients in the nodal basis expansion. The operator $\mathcal{I}_h' : \mathbb{R}^{\dim V_h} \to V_h'$ maps vectors to linear functionals $F \in V_h'$ given by

$$(\mathcal{I}_h' \boldsymbol{f})[u] = \boldsymbol{f}^T(\mathcal{I}_h^{-1}u), \quad \forall u \in V_h. \tag{14}$$

(i) Show that

$$A_h u = \mathcal{I}_h'(A\boldsymbol{u}), \quad \forall u \in V_h. \tag{15}$$

where A is the matrix corresponding to A_h and $\boldsymbol{u}$ is the vector of basis coefficients of u.

[4 marks]

(ii) Hence show that

$$A = (\mathcal{I}_h')^{-1} A_h \mathcal{I}_h. \tag{16}$$

[3 marks]

(c) Now consider a second bilinear form

$$b_h(u, v) = \int_\Omega uv + \nabla u \cdot \nabla v \, \mathrm{d}\, x, \tag{17}$$

with corresponding matrix B, and operator $B_h : V_h \to V_h'$.

(i) Show that

$$B^{-1}A = \mathcal{I}_h^{-1} B_h^{-1} A_h \mathcal{I}_h. \tag{18}$$

[4 marks]

(ii) Explain why $B_h^{-1}A_h$ has the same eigenvalues as $B^{-1}A$.

[3 marks]