Temporary front page

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite elements: numerical analysis and implementation

Date: ??

Time: ??

Time Allowed: 2 Hours

This paper has *4 Questions*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. What is the choice of the geometric decomposition (allocation of nodal variables to cell and vertex entities) that leads to the maximum possible global continuity of finite element spaces defined on the interval $[0, L]$ constructed from the following one-dimensional elements (K, P, N). Justify your answer.

(a) $K = [a, b]$, P is linear polynomials, $N = (N_1, N_2)$ where $N_1[u] = u((a + b)/2)$, $N_2[u] = u'((a + b)/2)$. **Solution:** ***SIMILAR***
Since the local space is 2-dimensional, a C^0 geometric decomposition would require one nodal variable allocated to each of the two vertices. We have $u(a) = N_1[u] - (a - b)N_2[u]/2$, and $u(b) = N_1[u] + (a - b)N_2[u]/2$. This means that both nodal variables are required to determine u at each end of the interval. This means it is not possible to allocate one nodal variable to each vertex such that the function value can only be determined from nodal variables associated with that vertex.

[6 marks]

(b) $K = [a, b]$, P is quadratic polynomials, $N = (N_1, N_2, N_3)$ where $N_1[u] = u(a)$, $N_2[u] = u(b)$, $N_3[u] = \int_a^b u \, \mathrm{d}\, x$. **Solution:** ***SIMILAR***
A C^1 geometric decomposition would require at least two nodal variables allocated to each vertex, so that both the function and the derivative can be determined, but the local space is 3 dimensional which does not give enough nodal variables. Hence the global space is at most C^0. A C^0 decomposition allocates N_1 to the vertex a, N_2 to the vertex b, and N_3 to the cell. Clearly $u(a)$ can be determined from N_1 and $u(b)$ from N_2 as required.

[6 marks]

(c) $K = [a, b]$, P is quadratic polynomials, $N = (N_1, N_2, N_3)$ where $N_1[u] = u'(a)$, $N_2[u] = u'(b)$,$N_3[u] = u((a + b)/2)$. **Solution:** ***SIMILAR***
The space must be at most C^0 by the arguments in the previous part. We have $u(a) = N_3[u] + (a - b)(N_1[u] + N_2[u])/4$, $u(b) = N_3[u] - (a - b)(N_1[u] + N_2[u])/4$, which means that all three variables are required to determine the function values at each vertex. By a similar argument to the first part, this means that a C^0 geometric decomposition is impossible.

[7 marks]

2. (a) Consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

- $*$ K is a non-degenerate triangle,
- $*$ $\mathcal{P}$ is the space of polynomials on K of degree ≤ 1.
- $*$ $\mathcal{N} = (N_1, N_2, N_3)$, where

$$N_i(u) = \int_{f_i} u \,\mathrm{d}\, x,$$

where (f_1, f_2, f_3) are the edges of K, with f_1 joining vertices 1 and 2, f_2 joining vertices 2 and 3, and f_3 joining vertices 3 and 1.

Show that $\mathcal{N}$ determines $\mathcal{P}$.

[10 marks]

Solution: ***SIMILAR***

It suffices to show that if $u \in P$, then $N_i(u) = 0$ for all $i \implies u = 0$.

So, we assume that $u \in P$ with $N_i(u) = 0$, looking to show that $u = 0$. $N_i(u) = 0$ means that the average of u over the edge f_i is zero. Since u is linear on f_i, this means that u vanishes at the midpoint of each edge. These edges can be joined by three lines L_1, L_2, L_3, and we iteratively conclude that u vanishes on L_1 and L_2, so that $u = cL_1(x)L_2(x)$, and u vanishing on the third vertex not intersected by L_1 means that $c = 0$ (following the usual argument for linear Lagrange elements on triangles), and hence $u = 0$ everywhere.

(b) Now consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

- $*$ K is a non-degenerate triangle,
- $*$ $\mathcal{P}$ is the space of polynomials on K of degree ≤ 2.
- $*$ $\mathcal{N} = (N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}, N_{3,1}, N_{3,2})$, where

$$N_{i,j}(u) = \int_{f_i} \phi_{i,j} u \,\mathrm{d}\, x,$$

where the edge test functions $\phi_{i,j}$ define a basis for linear functions restricted to f_i such that $\phi_{i,1} = 1$ on vertex 1 and 0 on vertex 2, $\phi_{i,2} = 1$ on vertex 2 and 0 on vertex 1, *etc.*

Show that $\mathcal{N}$ does not determine $\mathcal{P}$.

[10 marks]

Solution: ***UNSEEN***

We show by counter example. We take the quadratic q function that is equal to $1/6$ on each vertex, and $-1/12$ at each edge midpoint (this defines a unique quadratic function since these are the nodal variables for the standard Lagrange quadratic element, which is unisolvent). Consider one of the edges f_i, and choose a coordinate s which is equal to 0 on one end of the edge, and 1 on the other end. On that edge, $q|_{f_i}(s) = s^2 - s + 1/6$. This function has mean zero, and is symmetric, which means that

$$\int_{f_i} \phi(s) q|_{f_i}(s) \,\mathrm{d}\, s = 0$$

for any linear function $\phi(s)$, and hence $N_{i,j}(u) = 0$, $j = 1, 2$. This means that all the nodal variables vanish when applied to u, but u is not zero. Hence, $\mathcal{N}$ does not determine $\mathcal{P}$.

3. (a) Let b be a continuous, coercive bilinear form on V, and F be a continuous linear form on V. Let $u \in V$ solve the linear variational problem

$$b(u, v) = F(v) \quad \forall v \in V.$$

Let V_h be a finite dimensional subspace of V, and let $u_h \in V$ solve the Galerkin approximation

$$b(u_h, v) = F(v) \quad \forall v \in V_h.$$

Show that

$$b(u - u_h, v) = 0, \quad \forall v \in V_h.$$

[4 marks]

Solution: ***BOOKWORK***

Since $V_h \subset V$, we can take $v \in V_h$ in the variational problem for u, to get

$$b(u, v) = F(v) \quad \forall v \in V_h.$$

Then, subtracting the Galerkin approximation, we have

$$b(u, v) - b(u_h, v) = 0 \quad \forall v \in V_h.$$

Finally, from bilinearity, we have

$$b(u - u_h, v) = 0 \quad \forall v \in V_h.$$

(b) Hence, show that

$$\|u - u_h\|_V \leq \frac{M}{\gamma} \min_{v \in V_h} \|u - v\|_V,$$

where γ and M are the coercivity and continuity constants for b respectively.

[4 marks]

Solution: ***BOOKWORK***

$$\begin{aligned} \gamma \|u - u_h\|_V^2 &\leq b(u - u_h, u - u_h), \quad \textit{[coercivity]} \\ &= b(u - u_h, u - v + v - u_h) \quad \textit{for any } v \in V_h, \\ &= b(u - u_h, u - v) + \underbrace{b(u - u_h, v - u_h)}_{=0 \textit{[by previous part]}}, \quad \textit{[bilinearity]} \\ &\leq M \|u - u_h\|_V \|u - v\|_V. \quad \textit{[continuity]} \end{aligned}$$

Dividing both sides by $\gamma \|u - u_h\|_V$ gives

$$\|u - u_h\|_V \leq \frac{M}{\gamma} \|u - v\|_V.$$

Since the left-hand side is independent of the choice of v, we can minimise over $v \in V_h$ to get the result.

(c) Consider the variational problem of finding $u \in H^1([0,1])$ such that

$$\int_0^1 vu + v'u' \,\mathrm{d}\,x = \int_0^1 vx \,\mathrm{d}\,x + v(1) - v(0), \quad \forall v \in H^1([0,1]).$$

After dividing the interval $[0,1]$ into N equispaced cells and forming a $P1$ C^0 finite element space V_N, the error $\|u - u_h\|_{H^1} = 0$ for any $N > 0$.

Explain why this is expected.

[6 marks]

Solution: ***SIMILAR***

The problem has the solution $u(x) = x$. Hence, $u \in V_N$ for any N, and $\min_v \|v - u\|_{H^1([0,1])} = 0$, hence $\|u - u_h\|_{H^1([0,1])} = 0$ from the previous result.

(d) Let $\mathring{H}^1([0,1])$ be the subspace of $H^1([0,1])$ such that $u(0) = 0$. Consider the variational problem of finding $u \in \mathring{H}^1([0,1])$ with

$$\int_0^1 v'u' \,\mathrm{d}\,x = \int_0^{1/2} v \,\mathrm{d}\,x, \quad \forall v \in \mathring{H}([0,1]).$$

The interval $[0,1]$ is divided into $3N$ equispaced cells (where N is a positive integer). After forming a $P1$ C^0 finite element space V_N, the error $\|u - u_h\|_{H^1}$ is found not to converge to zero. Explain why this is expected?

[6 marks]

Solution: ***UNSEEN***

The solution is

$$u(x) = \begin{cases} \frac{x - x^2}{2} & x < 1/2, \\ \frac{1}{8} & \text{otherwise.} \end{cases}$$

This solution is not in H^2, and so the usual interpolation estimate is not expected.

4. The inhomogeneous Helmholtz equation in two dimensions is given by

$$\alpha(x)u - \nabla^2 u = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\partial\Omega$ is the boundary of the problem domain Ω, and $\alpha(x)$ is a $C^\infty(\Omega)$ function with bounds $1 \leq \alpha(x) \leq 2$ for all $z \in \Omega$.

(a) Write down a variational formulation for this problem, in the form

$$a(u, v) = F(v), \quad \forall v \in H^1(\Omega),$$

and show that if u solves the variational formulation, and $u \in H^2(\Omega)$ then u solves (1) in an appropriate sense.

[6 marks]

Solution: ***BOOKWORK***

We take

$$a(u, v) = \int_\Omega \alpha uv + \nabla u \cdot \nabla v \, \mathrm{d}\, x, \quad F(v) = \int_\Omega vf \, \mathrm{d}\, x.$$

Taking $v \in C_0^\infty(\Omega)$, we have enough regularity for integration by parts, and

$$\int_\Omega v(\alpha u - \nabla^2 u - f) \, \mathrm{d}\, x = 0.$$

Picking a sequence of vs that converge to $\alpha u - \nabla^2 u - f$, and passing to the limit gives $\|\alpha u - \nabla^2 u - f\|_{L^2(\Omega)}$, i.e. $\alpha u - \nabla^2 u = f$ in $L^2(\Omega)$. Returning to the variational form and using this fact gives

$$\begin{aligned} 0 &= \int_\Omega \alpha uv + \nabla u \cdot \nabla v - vf \, \mathrm{d}\, x, \\ &= \int_\Omega \alpha uv + \nabla u \cdot \nabla v - v(\alpha u - \nabla^2 u) \, \mathrm{d}\, x, \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}\, S, \quad \forall v \in H^1(\Omega), \end{aligned}$$

after using integration by parts. We may choose $v = \frac{\partial u}{\partial n}$ from the trace theorem, and hence $\|\frac{\partial u}{\partial n}\|_{L^2(\Omega)} = 0$ i.e. $\frac{\partial u}{\partial n} = 0$ in $L^2(\Omega)$.

(b) Show that $a(\cdot, \cdot)$ is continuous and coercive.

[6 marks]

Solution: ***SIMILAR***

$$|a(u, v)| = |\int_\Omega \alpha uv + \nabla u \cdot \nabla v \, \mathrm{d}\, x| \leq 2|\int_\Omega uv + \nabla u \cdot \nabla v \, \mathrm{d}\, x| = 2|(u, v)|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

by the Schwartz inequality. Hence a is continuous with continuity constant 2.

$$a(u, u) = \int_\Omega \alpha u^2 + |\nabla u|^2 \, \mathrm{d}\, x \geq \int_\Omega u^2 + |\nabla u|^2 \, \mathrm{d}\, x = \|u\|^2_{H^1(\Omega)},$$

hence a is coercive with coercivity constant 1.

(c) Hence, show that the linear Lagrange finite element approximation satisfies

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}.$$

for $C > 0$, independent of u. (You may make use of the approximation theory estimate

$$\|u - I_h u\|_{H^1(\Omega)} \leq \hat{C}h\|u\|_{H^2(\Omega)}.$$

for $\hat{C} > 0$, independent of u, where I_h is the nodal interpolation operator $I_h : H^2(\Omega) \to V_h$, where V_h is the finite element space with mesh parameter h, and any other results from lectures.)

[8 marks]

Solution: ***UNSEEN***

From Cèa's Lemma (also proven in Q3) we have

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{M = 2}{\gamma = 1} \min_{v \in V_h} \|v - u\|_{H^1(\Omega)} \leq 2\|I_h u - u\|_{H^1(\Omega)} \leq 2\hat{C}h\|u\|_{H^2(\Omega)}.$$